Analysis Problems and Solutions

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These problems come from Kenneth A. Ross's 2nd edition of "Elementary Analysis". I will say this more than once: sorry for typos!

Limits of Sequences

Problem 7.2

Determine if a sequence converges, if yes compute its limit.

(a)
$$s_n = \frac{1}{3n+1}, \quad \text{Converges to } 0.$$

(b)
$$b_n = \frac{3n+1}{4n-1}, \quad \text{Converges to } \frac{3}{4}.$$

(c)
$$c_n = \frac{n}{3^n}, \quad \text{Converges to } 0.$$

(d)
$$\sin(\frac{n\pi}{4}), \quad \text{Does not converge.}$$

Problem 7.4

(a) Give examples of a sequence of irrational numbers having a limit that IS a rational number.

$$x_n = \frac{\sqrt{2}}{3n}, \quad \lim_{n\to\infty} x_n = 0$$

(b) Give a sequence of rational numbers that has a limit being an irrational number.

$$r_n = \left(1 + \frac{1}{n}\right)^n, \quad \lim_{n\to\infty} r_n = e$$

Problem 8.2

Determine the limits of the following sequences then prove claims.

(a)

$$a_n = \frac{n}{n^2+1} \to 0$$

Proof. Let $\varepsilon > 0$ and $N = \frac{1}{\epsilon}$. Then for $n > N$ we have

$$\left|\frac{n}{n^2+1} - 0\right| = \left|\frac{n}{n^2+1}\right| < \left|\frac{1}{n}\right| < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} a_n = 0$. □

(b)

$$b_n = \frac{7n-19}{3n+7} \to \frac{7}{3}$$

Proof. Let $\varepsilon > 0$ and $N = \frac{106}{9\varepsilon}$, Then for $n > N$ we have

$$\left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| = \left|\frac{106}{9n+21}\right| < \left|\frac{106}{9n}\right| < \left|\frac{106}{9N}\right| = \left|\frac{106}{9(\frac{106}{9\varepsilon})}\right| = \varepsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} b_n = \frac{7}{3}$. □

(c)

$$c_n = \frac{4n+3}{7n-5} \to \frac{4}{7}$$

Proof. Let $\varepsilon > 0$ and $N = \max\left\{1, \frac{41}{14\varepsilon}\right\}$. Then for $n > N$ we have

$$\left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| = \left|\frac{41}{49n-35}\right| < \left|\frac{41}{14n}\right| < \left|\frac{41}{14N}\right| = \left|\frac{41}{14(\frac{41}{14\varepsilon})}\right| \leq \varepsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} c_n = \frac{4}{7}$. □

(d)

$$d_n = \frac{2n+4}{5n+2} \to \frac{2}{5}$$

Proof. Let $\varepsilon > 0$ and $N = \frac{16}{25\varepsilon}$. Then for $n > N$ we have

$$\left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| = \left|\frac{16}{25n+10}\right| < \left|\frac{16}{25n}\right| < \left|\frac{16}{25N}\right| = \left|\frac{16}{25(\frac{16}{25\varepsilon})}\right| = \varepsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} d_n = \frac{2}{5}$. □

(e)

$$s_n = \frac{1}{n}\sin(n) \to 0$$

Proof. Let $\varepsilon > 0$ and $N = \frac{1}{\varepsilon}$. Then for $n > N$ we have

$$\left|\frac{1}{n}\sin(n) - 0\right| = \left|\frac{1}{n}\sin(n)\right| \leq \left|\frac{1}{n}\right| < \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} s_n = 0$ □

Problem 8.4

Let (t_n) be a bounded sequence i.e. there exists an M such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove $\lim(s_n t_n) = 0$.

Proof. Let $\varepsilon > 0$. Since $(s_n) \to 0$, there exists an N such that for $n > N$, $|s_n| < \frac{\varepsilon}{M}$. So for $M > 0$ and $n > N$ we have

$$|s_n t_n| = |s_n||t_n| \leq |s_n|M < \frac{\varepsilon}{M}M = \varepsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} s_n t_n = 0$. □

Problem 8.8

Prove the following statements.

(a)

$$\lim[\sqrt{n^2+1} - n] = 0$$

Proof.

$$\lim[\sqrt{n^2+1}-n] = \lim\left[\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+1)}\right] = \lim\left[\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}\right] = \lim\left[\frac{1}{\sqrt{n^2+1}+n}\right] = \boxed{0}.$$

□

(b)

$$\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$$

Proof.

$$\lim[\sqrt{n^2+n}-n] = \lim\left[\frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{(\sqrt{n^2+n}+n)}\right] = \lim\left[\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}\right] = \lim\left[\frac{n}{\sqrt{n^2+n}+n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right] = \frac{1}{\sqrt{1+0}+1} = \boxed{\frac{1}{2}}$$

□

(c)

$$\lim[\sqrt{4n^2+n} - 2n] = \frac{1}{4}$$

Proof.

$$\lim[\sqrt{4n^2+n}-2n] = \lim\left[\frac{(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n)}{(\sqrt{4n^2+n}+2n)}\right] = \lim\left[\frac{4n^2+n-4n^2}{\sqrt{4n^2+n}-2n}\right] = \lim\left[\frac{n}{\sqrt{4n^2+n}+2n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{4+\frac{1}{n}}+2}\right] = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

□

Problem 9.4

Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) List the first four terms of (s_n).

$$\begin{aligned} s_1 &= 1 \\ s_2 &= \sqrt{s_1 + 1} = \sqrt{2} \\ s_3 &= \sqrt{s_2 + 1} = \sqrt{\sqrt{2} + 1} \\ s_4 &= \sqrt{s_3 + 1} = \sqrt{\sqrt{\sqrt{2} + 1} + 1} \end{aligned}$$

(b) It turns out that (s_n) converges. Assume this fact and prove the limit is $\frac{1}{2}(1 + \sqrt{5})$.

Proof. Since (s_n) converges, let

$$\lim_{n\to\infty} s_n = L.$$

Since $s_{n+1} = \sqrt{s_n + 1}$, taking limits on both sides gives

$$L = \sqrt{L + 1}.$$

Squaring both sides gives

$$L^2 = L + 1.$$

Thus

$$L^2 - L - 1 = 0.$$

Using the quadratic formula, we get

$$L = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

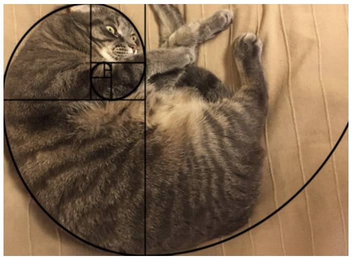
Since $s_n > 0$ for all n, we know $L \geq 0$. Therefore,

$$L = \frac{1 + \sqrt{5}}{2}.$$

Hence,

$$\lim_{n\to\infty} s_n = \frac{1}{2}(1 + \sqrt{5}).$$

□



Functional Limits and Continuity

Problem 17.4

Prove the function $\sqrt{x}$ is continuous on its domain $[0, \infty)$.

Proof. Let $a \in [0, \infty)$ and let $\varepsilon > 0$. We want to show that there exists a $\delta > 0$ such that if

$$|x - a| < \delta,$$

then

$$|\sqrt{x} - \sqrt{a}| < \varepsilon.$$

We consider two cases.
Case I: $a > 0$. Let

$$\delta = \sqrt{a}\varepsilon.$$

Then if $|x - a| < \delta$, we have

$$|\sqrt{x} - \sqrt{a}| = \left| \frac{x - a}{\sqrt{x} + \sqrt{a}} \right|.$$

Since $x \geq 0$, we know $\sqrt{x} \geq 0$. Therefore,

$$\sqrt{x} + \sqrt{a} \geq \sqrt{a}.$$

So

$$|\sqrt{x} - \sqrt{a}| = \left| \frac{x - a}{\sqrt{x} + \sqrt{a}} \right| \leq \frac{|x - a|}{\sqrt{a}} < \frac{\delta}{\sqrt{a}} = \frac{\sqrt{a}\varepsilon}{\sqrt{a}} = \varepsilon.$$

Case II: $a = 0$. Let

$$\delta = \varepsilon^2.$$

Then if $|x - 0| < \delta$, we have

$$|x| < \varepsilon^2.$$

Since $x \in [0, \infty)$, this gives

$$0 \leq x < \varepsilon^2.$$

Taking square roots,

$$0 \leq \sqrt{x} < \varepsilon.$$

Hence,

$$|\sqrt{x} - \sqrt{0}| = \sqrt{x} < \varepsilon.$$

Therefore, $\sqrt{x}$ is continuous at every $a \in [0, \infty)$. Hence, $\sqrt{x}$ is continuous on its domain $[0, \infty)$. □

Problem 17.10

Prove the following functions are discontinuous at the indicated points.

(a)

$$f(x) = 1 \text{ for } x > 0 \quad \text{and} \quad f(x) = 0 \text{ for } x \leq 0, \quad x_0 = 0.$$

Proof. We will show that f is not continuous at 0.
Let

$$x_n = \frac{1}{n}.$$

Then $x_n > 0$ for all n, and

$$x_n \to 0.$$

However, since $x_n > 0$, we have

$$f(x_n) = 1$$

for all n. Therefore,

$$\lim_{n\to\infty} f(x_n) = 1.$$

But

$$f(0) = 0.$$

Thus,

$$\lim_{n\to\infty} f(x_n) \neq f(0).$$

Therefore, f is discontinuous at $x_0 = 0$. □

(b)

$$g(x) = \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0 \quad \text{and} \quad g(0) = 0, \quad x_0 = 0.$$

Proof. We will show that g is not continuous at 0.

Let

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}.$$

Then

$$x_n \to 0.$$

However,

$$g(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1.$$

Therefore,

$$\lim_{n\to\infty} g(x_n) = 1.$$

But

$$g(0) = 0.$$

Thus,

$$\lim_{n\to\infty} g(x_n) \neq g(0).$$

Therefore, g is discontinuous at $x_0 = 0$. □

(c)

$$\operatorname{sgn}(x) = -1 \text{ for } x < 0, \quad \operatorname{sgn}(x) = 1 \text{ for } x > 0, \quad \operatorname{sgn}(0) = 0, \quad x_0 = 0.$$

Proof. We will show that $\operatorname{sgn}(x)$ is not continuous at 0.

Let

$$x_n = \frac{1}{n}.$$

Then $x_n > 0$ for all n, and

$$x_n \to 0.$$

Since $x_n > 0$, we have

$$\operatorname{sgn}(x_n) = 1$$

for all n. Therefore,

$$\lim_{n\to\infty} \operatorname{sgn}(x_n) = 1.$$

But

$$\operatorname{sgn}(0) = 0.$$

Thus,

$$\lim_{n\to\infty} \operatorname{sgn}(x_n) \neq \operatorname{sgn}(0).$$

Therefore, $\operatorname{sgn}(x)$ is discontinuous at $x_0 = 0$. □

Problem 18.6

Prove $x = \cos x$ for some x in $\left(0, \frac{\pi}{2}\right)$.

Proof. Let

$$f(x) = x - \cos x.$$

Since x and $\cos x$ are continuous functions, $f(x)$ is continuous on

$$\left[0, \frac{\pi}{2}\right].$$

Now we evaluate f at the endpoints:

$$f(0) = 0 - \cos(0) = -1 < 0,$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} > 0.$$

Therefore,

$$f(0) < 0 < f\left(\frac{\pi}{2}\right).$$

Since f is continuous on $\left[0, \frac{\pi}{2}\right]$, by the Intermediate Value Theorem, there exists some

$$c \in \left(0, \frac{\pi}{2}\right)$$

such that

$$f(c) = 0.$$

Thus,

$$c - \cos c = 0.$$

Hence,

$$c = \cos c.$$

Therefore, there exists some $x \in \left(0, \frac{\pi}{2}\right)$ such that

$$x = \cos x.$$

□

Problem 19.4

(a) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.

Proof. Assume, for contradiction, that f is not bounded on S.
Then for each $n \in \mathbb{N}$, there exists some $x_n \in S$ such that

$$|f(x_n)| > n.$$

Since S is bounded, the sequence (x_n) is bounded. Therefore, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence

$$(x_{n_k})$$

of (x_n). Since every convergent sequence is Cauchy, (x_{n_k}) is a Cauchy sequence.
Since f is uniformly continuous on S, it follows that

$$(f(x_{n_k}))$$

is also a Cauchy sequence. Therefore, $(f(x_{n_k}))$ is bounded.
But our setup requirs that,

$$|f(x_{n_k})| > n_k.$$

Since $n_k \to \infty$, this means $(f(x_{n_k}))$ is unbounded, which is a contradiction.
Therefore, f must be bounded on S. □

(b) Use (a) to give yet another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

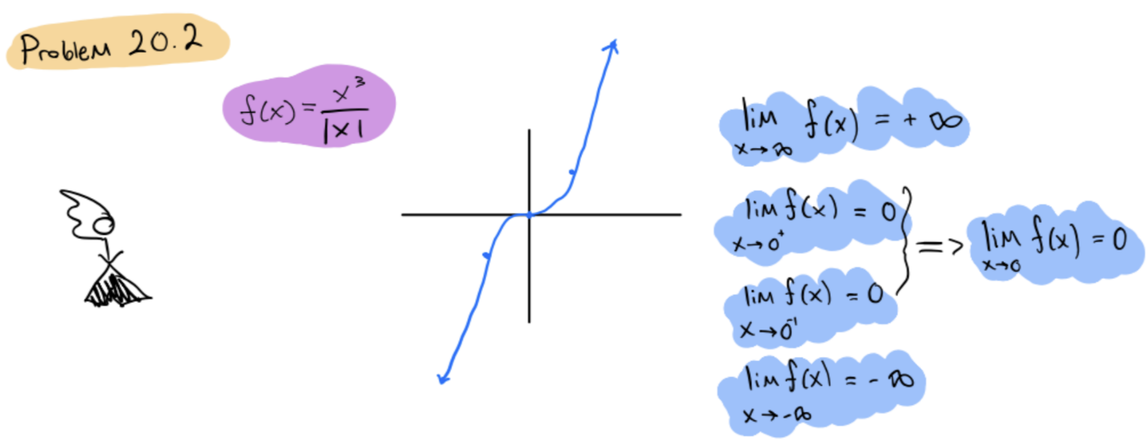
Proof. Let
$$f(x) = \frac{1}{x^2}.$$
The set $(0, 1)$ is bounded.

However, f is not bounded on $(0, 1)$. To see this, let
$$x_n = \frac{1}{n}.$$
Then $x_n \in (0, 1)$ for $n > 1$, and
$$f(x_n) = \frac{1}{\left(\frac{1}{n}\right)^2} = n^2.$$
Since
$$n^2 \to \infty,$$
the function $f(x) = \frac{1}{x^2}$ is unbounded on $(0, 1)$.

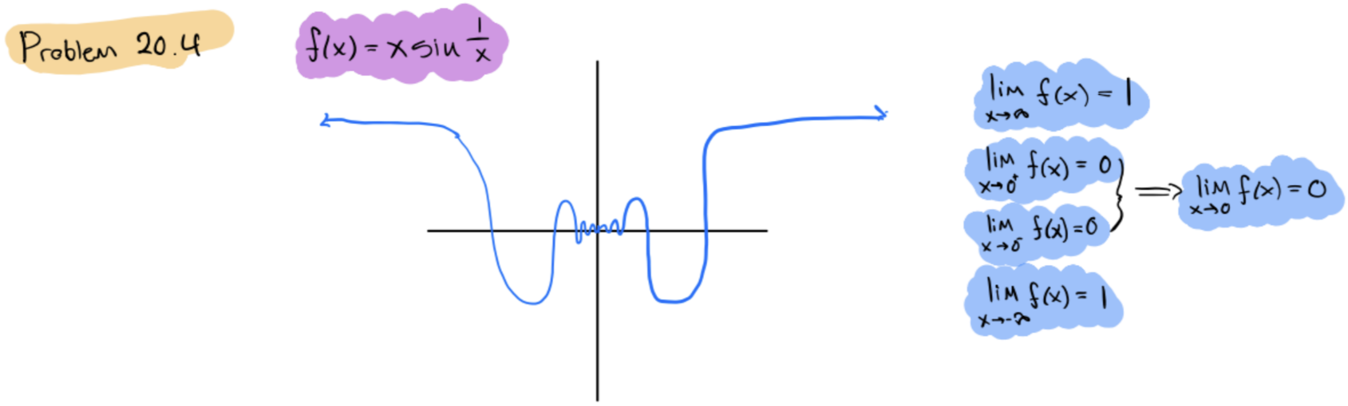
By part (a), if f were uniformly continuous on the bounded set $(0, 1)$, then f would have to be bounded on $(0, 1)$. But f is not bounded on $(0, 1)$.

Therefore,
$$f(x) = \frac{1}{x^2}$$
is not uniformly continuous on $(0, 1)$. □

Problem 20.2



Problem 20.4



Differentiation

Problem 28.2

Use the definition of derivative to calculate the derivatives of the following functions at the indicated points.

(a)

$$f(x) = x^3 \quad \text{at } x = 2.$$

Proof.

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{x^3 - 8}{x - 2}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2 + 2x + 4)}{x - 2} = \lim_{x\to 2}(x^2 + 2x + 4) = 12.$$

Therefore,

$$\boxed{f'(2) = 12}.$$

□

(b)

$$g(x) = x + 2 \quad \text{at } x = a.$$

Proof.

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a} = \lim_{x\to a} \frac{(x+2) - (a+2)}{x - a}$$

$$= \lim_{x\to a} \frac{x - a}{x - a} = \lim_{x\to a} 1 = 1.$$

Therefore,

$$\boxed{g'(a) = 1}.$$

□

(c)

$$f(x) = x^2 \cos x \quad \text{at } x = 0.$$

Proof.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \cos x - 0}{x}$$
$$= \lim_{x\to 0} x \cos x = 0.$$

Therefore,

$$\boxed{f'(0) = 0}.$$

□

(d)

$$r(x) = \frac{3x + 4}{2x - 1} \quad \text{at } x = 1.$$

Proof.

$$r'(1) = \lim_{x\to 1} \frac{r(x) - r(1)}{x - 1} = \lim_{x\to 1} \frac{\frac{3x+4}{2x-1} - 7}{x - 1}$$
$$= \lim_{x\to 1} \frac{\frac{3x+4-7(2x-1)}{2x-1}}{x - 1} = \lim_{x\to 1} \frac{\frac{-11x+11}{2x-1}}{x - 1}$$
$$= \lim_{x\to 1} \frac{\frac{-11(x-1)}{2x-1}}{x - 1} = \lim_{x\to 1} \frac{-11}{2x - 1} = -11.$$

Therefore,

$$\boxed{r'(1) = -11}.$$

□

Problem 28.6

Let $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Observe f is continuous at $x = 0$ by Exercise 17.9(c). (We can also visualy confirm continuity by looking at problem 20.4!)

(b) Is f differentiable at $x = 0$? Justify your answer.

Proof. Using the definition of derivative,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}.$$

Since $f(0) = 0$, we get

$$f'(0) = \lim_{x\to 0} \frac{x \sin \frac{1}{x}}{x}.$$

Thus,

$$f'(0) = \lim_{x\to 0} \sin \frac{1}{x}.$$

However,

$$\lim_{x\to 0} \sin \frac{1}{x}$$

does not exist.

Therefore, $f'(0)$ does not exist, so f is not differentiable at $x = 0$. □

Problem 28.10

Let

$$h(x) = (\cos x + e^x)^{12}.$$

(a) Calculate $h'(x)$.

Proof. Using the chain rule,

$$h'(x) = 12(\cos x + e^x)^{11} \frac{d}{dx}(\cos x + e^x).$$

Since

$$\frac{d}{dx}(\cos x + e^x) = -\sin x + e^x,$$

we get

$$\boxed{h'(x) = 12(\cos x + e^x)^{11}(-\sin x + e^x)}.$$

□

(b) Show how the chain rule justifies your computation in part (a) by writing $h = g \circ f$ for suitable f and g.

Proof. Let

$$f(x) = \cos x + e^x$$

and

$$g(x) = x^{12}.$$

Then

$$h(x) = g(f(x)) = (g \circ f)(x).$$

Now

$$f'(x) = -\sin x + e^x$$

and

$$g'(x) = 12x^{11}.$$

Therefore, by the chain rule,

$$h'(x) = g'(f(x))f'(x).$$

Thus,

$$h'(x) = 12(\cos x + e^x)^{11}(-\sin x + e^x).$$

□

Problem 29.2

Prove $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$. Since $\cos x$ is differentiable on $\mathbb{R}$, by the Mean Value Theorem there exists some c between x and y such that

$$\cos x - \cos y = -\sin(c)(x - y).$$

Taking absolute values gives

$$|\cos x - \cos y| = |-\sin(c)||x - y|.$$

Since

$$|\sin(c)| \leq 1,$$

we have

$$|\cos x - \cos y| \leq |x - y|.$$

Therefore,

$$\boxed{|\cos x - \cos y| \leq |x - y|}.$$

□

Problem 29.6

Give the equation of the straight line used in the proof of the Mean Value Theorem.

Proof. The straight line used in the proof of the Mean Value Theorem is the line through the points

$$(a, f(a)) \quad \text{and} \quad (b, f(b)).$$

Its slope is

$$\frac{f(b) - f(a)}{b - a}.$$

Therefore, the equation of the line is

$$\boxed{y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)}.$$

□

Problem 30.2

Find the following limits if they exist.

(a)

$$\lim_{x \to 0} \frac{x^3}{\sin x - x}$$

By L'Hopital's Rule,

$$\lim_{x \to 0} \frac{x^3}{\sin x - x} = \lim_{x \to 0} \frac{3x^2}{\cos x - 1} = \lim_{x \to 0} \frac{6x}{-\sin x} = \lim_{x \to 0} \frac{6}{-\cos x} = \boxed{-6}.$$

(b)

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

By L'Hopital's Rule,

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x}$$

$$= \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x} = \lim_{x \to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}.$$

(c)

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x}.$$

By L'Hopital's Rule,

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = \boxed{0}.$$

(d)

$$\lim_{x \to 0} (\cos x)^{1/x^2}$$

Let

$$L = \lim_{x \to 0} (\cos x)^{1/x^2}.$$

Then

$$\ln L = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2}.$$

By L'Hopital's Rule,

$$\ln L = \lim_{x\to 0} \frac{-\tan x}{2x} = \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}.$$

Therefore,

$$\boxed{L = e^{-1/2}}.$$

Integration

Problem 32.2

Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

(a) Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.

Solution. Let

$$P = \{0 = x_0 < x_1 < \cdots < x_n = b\}$$

be a partition of $[0, b]$. Since every interval contains rational and irrational numbers,

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = x_i$$

and

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0.$$

Thus

$$U(f, P) = \sum_{i=1}^{n} x_i(x_i - x_{i-1})$$

and

$$L(f, P) = \sum_{i=1}^{n} 0(x_i - x_{i-1}) = 0.$$

Therefore,

$$\underline{\int_0^b} f = 0.$$

Also, the upper sums are the same as upper sums for the function $y = x$ on $[0, b]$, so

$$\overline{\int_0^b} f = \int_0^b x\,dx = \frac{b^2}{2}.$$

Hence,

$$\boxed{\underline{\int_0^b} f = 0} \quad \text{and} \quad \boxed{\overline{\int_0^b} f = \frac{b^2}{2}}.$$

□

(b) Is f integrable on $[0, b]$?

Solution. Since

$$\underline{\int_0^b} f = 0 \quad \text{and} \quad \overline{\int_0^b} f = \frac{b^2}{2},$$

the upper and lower Darboux integrals are not equal for $b > 0$.

Therefore,

$$\boxed{f \text{ is not integrable on } [0, b].}$$

□

Problem 32.6

Let f be a bounded function on $[a, b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that

$$\lim_{n\to\infty} (U_n - L_n) = 0.$$

Show f is integrable and

$$\int_a^b f = \lim U_n = \lim L_n.$$

Proof. Since

$$\lim_{n\to\infty} (U_n - L_n) = 0,$$

for every $\varepsilon > 0$, there exists N such that for $n > N$,

$$U_n - L_n < \varepsilon.$$

Therefore, by Theorem 32.5, f is integrable on $[a, b]$.

Let

$$I = \int_a^b f.$$

Since U_n is an upper Darboux sum and L_n is a lower Darboux sum,

$$L_n \leq I \leq U_n.$$

Thus,

$$0 \leq U_n - I \leq U_n - L_n$$

and

$$0 \leq I - L_n \leq U_n - L_n.$$

Since

$$\lim_{n\to\infty} (U_n - L_n) = 0,$$

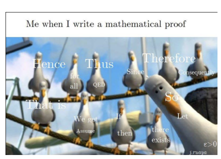
we get

$$\lim U_n = I \quad \text{and} \quad \lim L_n = I.$$

Therefore,

$$\boxed{\int_a^b f = \lim U_n = \lim L_n}.$$

□



Problem 33.4

Give an example of a function f on $[0, 1]$ that is not integrable for which $|f|$ is integrable.

Solution. We can define the function

$$f(x) = \begin{cases} 1 & \text{if x is rational} \\ -1 & \text{if x is irrational} \end{cases}$$

f on its own is not integrable because its always alternating between 1 and -1 in any interval, but $|f|$ is integrable because it is essentially a constant function equal to 1 everywhere! □

Problem 33.8

Let f and g be integrable functions on $[a, b]$.

(a) Show fg is integrable on $[a, b]$.

Proof. Since f and g are integrable on $[a, b]$, we know

$$f + g \quad \text{and} \quad f - g$$

are integrable on $[a, b]$.
By Exercise 33.7,

$$(f + g)^2 \quad \text{and} \quad (f - g)^2$$

are integrable on $[a, b]$.
Since

$$4fg = (f + g)^2 - (f - g)^2,$$

we have

$$fg = \frac{1}{4}\left[(f + g)^2 - (f - g)^2\right].$$

Therefore, fg is integrable on $[a, b]$. □

(b) Show $\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$.

Proof. Since f and g are integrable, we know

$$f + g \quad \text{and} \quad f - g$$

are integrable.
Since $f - g$ is integrable, $|f - g|$ is integrable.
By Exercise 17.8,

$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|.$$

Therefore, $\min(f, g)$ is integrable.
Also,

$$\max(f, g) = -\min(-f, -g).$$

Since $-f$ and $-g$ are integrable, $\min(-f, -g)$ is integrable. Therefore,

$$\max(f, g)$$

is integrable. □

Problem 34.2

Calculate the following limits.

(a)

$$\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt$$

Let

$$F(x) = \int_0^x e^{t^2}\, dt.$$

Then $F(0) = 0$, so

$$\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt = \lim_{x\to 0} \frac{F(x) - F(0)}{x - 0} = F'(0).$$

By the Fundamental Theorem of Calculus,

$$F'(x) = e^{x^2}.$$

Therefore,

$$F'(0) = e^0 = \boxed{1}.$$

(b)

$$\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt$$

Let

$$F(x) = \int_3^x e^{t^2}\, dt.$$

Then $F(3) = 0$, so

$$\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt = \lim_{h\to 0} \frac{F(3+h) - F(3)}{h} = F'(3).$$

By the Fundamental Theorem of Calculus,

$$F'(x) = e^{x^2}.$$

Therefore,

$$F'(3) = \boxed{e^9}.$$

Problem 34.6

Let f be a continuous function on $\mathbb{R}$ and define

$$G(x) = \int_0^{\sin x} f(t)\, dt$$

for $x \in \mathbb{R}$. Show G is differentiable on $\mathbb{R}$ and compute G'.

Proof. Let

$$F(u) = \int_0^u f(t)\, dt.$$

Then

$$G(x) = F(\sin x).$$

Since f is continuous, by the Fundamental Theorem of Calculus,

$$F'(u) = f(u).$$

Therefore, by the chain rule,

$$G'(x) = F'(\sin x) \cos x.$$

Thus,

$$\boxed{G'(x) = f(\sin x) \cos x}.$$

Hence, G is differentiable on $\mathbb{R}$. □

Sequences and Functions

Problem 23.2

Find the radius of convergence and determine the exact interval of convergence.

(a)

$$\sum \sqrt{n} x^n$$

Proof. By the ratio test,

$$\lim_{n\to\infty}\left|\frac{\sqrt{n+1}x^{n+1}}{\sqrt{n}x^n}\right| = |x|\lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = |x|.$$

So $R = 1$. At $x = 1$, $\sum\sqrt{n}$ diverges. At $x = -1$, $\sum(-1)^n\sqrt{n}$ diverges since the terms do not go to 0.

$$\boxed{R = 1, \quad I = (-1,1)}.$$

□

(b)

$$\sum\frac{1}{n\sqrt{n}}x^n$$

Proof. By the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)\sqrt{n+1}}}{\frac{x^n}{n\sqrt{n}}}\right| = |x|\lim_{n\to\infty}\frac{n\sqrt{n}}{(n+1)\sqrt{n+1}} = |x|.$$

So $R = 1$. At $x = 1$, $\sum\frac{1}{n^{3/2}}$ converges. At $x = -1$, $\sum\frac{(-1)^n}{n^{3/2}}$ converges absolutely.

$$\boxed{R = 1, \quad I = [-1,1]}.$$

□

(c)

$$\sum x^{n!}$$

Proof. By the ratio test,

$$\lim_{n\to\infty}\left|\frac{x^{(n+1)!}}{x^{n!}}\right| = \lim_{n\to\infty}|x|^{(n+1)!-n!}.$$

This is 0 if $|x| < 1$ and diverges if $|x| > 1$. So $R = 1$. At $x = 1$, $\sum 1$ diverges. At $x = -1$, $\sum(-1)^{n!}$ diverges since $(-1)^{n!} = 1$ for $n \geq 2$.

$$\boxed{R = 1, \quad I = (-1,1)}.$$

□

(d)

$$\sum\frac{3^n}{\sqrt{n}}x^{2n+1}$$

Proof. By the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{3^{n+1}}{\sqrt{n+1}}x^{2n+3}}{\frac{3^n}{\sqrt{n}}x^{2n+1}}\right| = 3|x|^2\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 3|x|^2.$$

So

$$3|x|^2 < 1 \implies |x| < \frac{1}{\sqrt{3}},$$

and

$$R = \frac{1}{\sqrt{3}}.$$

At $x = \frac{1}{\sqrt{3}}$, $\sum\frac{1}{\sqrt{3n}}$ diverges. At $x = -\frac{1}{\sqrt{3}}$, $\sum-\frac{1}{\sqrt{3n}}$ diverges.

$$\boxed{R = \frac{1}{\sqrt{3}}, \quad I = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}.$$

□

Problem 23.6

(a) Suppose $\sum a_n x^n$ has finite radius of convergence R and $a_n \geq 0$ for all n. Show that if the series converges at R, then it also converges at $-R$.

Proof. Since the series converges at R, we know

$$\sum a_n R^n$$

converges.

At $-R$, we have

$$\sum a_n(-R)^n = \sum (-1)^n a_n R^n.$$

Since

$$|(-1)^n a_n R^n| = a_n R^n,$$

and

$$\sum a_n R^n$$

converges, the series

$$\sum a_n(-R)^n$$

converges absolutely. Therefore, it converges at $-R$. □

(b) Give an example of a power series whose interval of convergence is exactly $[-1, 1)$.

Solution. Consider

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

By the ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|.$$

Thus $R = 1$.

Checkong endpoints we see:

$$x = 1 : \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$x = -1 : \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges.}$$

Therefore, the interval of convergence is

$$\boxed{[-1, 1)}.$$

□

Problem 23.8

For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{1}{n}\sin(nx).$$

Each f_n is differentiable. Show the following.

(a) Show $\lim f_n(x) = 0$ for all $x \in \mathbb{R}$.

Proof. Since

$$-1 \le \sin(nx) \le 1,$$

we have

$$-\frac{1}{n} \le \frac{1}{n}\sin(nx) \le \frac{1}{n}.$$

Since

$$\lim_{n\to\infty} -\frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0,$$

by the Squeeze Theorem,

$$\lim_{n\to\infty} f_n(x) = 0.$$

□

(b) But $\lim f_n'(x)$ need not exist.

Proof. First,

$$f_n'(x) = \cos(nx).$$

At $x = \pi$,

$$f_n'(\pi) = \cos(n\pi) = (-1)^n.$$

Since

$$(-1)^n$$

does not converge, we have

$$\lim_{n\to\infty} f_n'(\pi)$$

does not exist. □

Problem 24.2

For $x \in [0, \infty)$, let

$$f_n(x) = \frac{x}{n}.$$

(a) Find $f(x) = \lim f_n(x)$.

Proof. For fixed $x \in [0, \infty)$,

$$f(x) = \lim_{n\to\infty} \frac{x}{n} = 0.$$

Therefore,

$$\boxed{f(x) = 0}.$$

□

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

Proof. Let $\varepsilon > 0$ and choose

$$N = \frac{1}{\varepsilon}.$$

Then for $n > N$ and all $x \in [0, 1]$,

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{x}{n} \le \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

Therefore,

$$\boxed{f_n \to f \text{ uniformly on } [0, 1]}.$$

□

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Proof. We show f_n does not converge uniformly to f on $[0, \infty)$.
Let

$$\varepsilon = \frac{1}{2}.$$

For every N, choose $n > N$ and let

$$x = n.$$

Then

$$|f_n(x) - f(x)| = \left|\frac{n}{n} - 0\right| = 1 > \frac{1}{2}.$$

Therefore,

$$\boxed{f_n \not\to f \text{ uniformly on } [0, \infty)}.$$

□

Problem 24.4

For $x \in [0, \infty)$, let

$$f_n(x) = \frac{x^n}{1 + x^n}.$$

(a) Find $f(x) = \lim f_n(x)$.

Proof. For fixed $x \in [0, \infty)$,

$$0 \le x < 1 \implies \lim_{n\to\infty} \frac{x^n}{1 + x^n} = 0,$$
$$x = 1 \implies \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2},$$

and

$$x > 1 \implies \lim_{n\to\infty} \frac{x^n}{1 + x^n} = 1.$$

Therefore,

$$\boxed{f(x) = \begin{cases} 0, & 0 \le x < 1, \\ \frac{1}{2}, & x = 1, \\ 1, & x > 1. \end{cases}}$$

□

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

Proof. We show f_n does not converge uniformly on $[0, 1]$.
Let

$$\varepsilon = \frac{1}{4}.$$

For any N, choose $n > N$ and let

$$x = \left(\frac{1}{2}\right)^{1/n}.$$

Then $x \in [0, 1)$, so $f(x) = 0$. Also,

$$f_n(x) = \frac{x^n}{1 + x^n} = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}.$$

Thus,

$$|f_n(x) - f(x)| = \frac{1}{3} > \frac{1}{4}.$$

Therefore,

$$\boxed{f_n \not\to f \text{ uniformly on } [0,1]}.$$

□

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Proof. Since f_n does not converge uniformly on $[0, 1]$, it also does not converge uniformly on the larger interval $[0, \infty)$.

Therefore,

$$\boxed{f_n \not\to f \text{ uniformly on } [0,\infty)}.$$

□

Problem 26.4

(a) Observe

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

for $x \in \mathbb{R}$, since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Proof. Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

replace x with $-x^2$:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

□

(b) Express

$$F(x) = \int_0^x e^{-t^2}\, dt$$

as a power series.

Proof. From part (a),

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}.$$

Thus,

$$F(x) = \int_0^x e^{-t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}\, dt.$$

Therefore,

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}.$$

Hence,

$$\boxed{F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}}.$$

□

Sorry if there are typos!!!