Q-binomial Theorem Proof of Jacobi's Triple Product Identity

Ilan Shlain

Proof. We begin with the finite q-analog of the binomial theorem and its expansion:

$$(1+x)(1+xq)\dots(1+xq^{n-1}) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} x^k. \tag{1}$$

The q-binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

We now perform the substitution $x \mapsto -x$:

$$(1-x)(1-xq)\dots(1-xq^{n-1}) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} (-x)^k. \tag{2}$$

Next, we apply $n \mapsto 2n$ to expose a symmetry in the product:

$$(1-x)(1-xq)\dots(1-xq^{2n-1}) = \sum_{k=0}^{2n} \binom{2n}{k}_q q^{\frac{k(k-1)}{2}} (-x)^k. \tag{3}$$

We now work with the left-hand side:

$$(1-x)(1-xq)\dots(1-xq^{2n-1}) = \underbrace{(1-x)(1-xq)\dots(1-xq^{n-1})}_{n\text{-terms}}\underbrace{(1-xq^n)\dots(1-xq^{2n-1})}_{n\text{-terms}}. \tag{4}$$

We can factor out $(-x)^n$ from the first half:

$$= (-x)^n(-x^{-1}+1)(-x^{-1}+q)\dots(-x^{-1}+q^{n-1})(1-xq^n)\dots(1-xq^{2n-1}) \tag{5}$$

$$= (-x)^n(1-x^{-1})(q-x^{-1})\dots(q^{n-1}-x^{-1})(1-xq^n)\dots(1-xq^{2n-1}). \tag{6}$$

Next, we factor out the powers of q from each term of the form $q^j - x^{-1}$ for $j = 0, 1, \dots, n-1$. Since $\sum_{j=0}^{n-1} j = \frac{n(n-1)}{2}$, we obtain:

$$= (-x)^n q^{\frac{n(n-1)}{2}}(1-x^{-1})(1-x^{-1}q^{-1})\dots(1-x^{-1}q^{-n+1})(1-xq^n)\dots(1-xq^{2n-1}). \tag{7}$$

We then make another substitution $x \mapsto \frac{x}{q^n}$:

$$= (-x)^n q^{-\frac{n(n+1)}{2}}(1-x^{-1}q^n)(1-x^{-1}q^{n-1})\dots(1-x^{-1}q)(1-x)\dots(1-xq^{n-1}). \tag{8}$$

Thus, the left-hand side has been simplified to a symmetric product form.
Returning to the full equation:

$$(-x)^n q^{-\frac{n(n+1)}{2}}(1-x^{-1}q^n)(1-x^{-1}q^{n-1})\dots(1-x^{-1}q)(1-x)\dots(1-xq^{n-1}) \tag{9}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k}_q q^{\frac{k(k-1)}{2}} \left(\frac{-x}{q^n}\right)^k. \tag{10}$$

Simplifying the powers of q on the right-hand side gives:

$$\frac{k(k-1)}{2} + \frac{n(n+1)}{2} - nk = \frac{(k-n)(k-n-1)}{2}.$$

Hence:

$$(1-x^{-1}q^n)(1-x^{-1}q^{n-1})\dots(1-x^{-1}q)(1-x)\dots(1-xq^{n-1}) = \sum_{k=0}^{2n}\binom{2n}{k}_q q^{\frac{(k-n)(k-n-1)}{2}}(-x)^{k-n}. \tag{11}$$

Performing a change of index $l = k - n$ gives:

$$(1-x^{-1}q^n)(1-x^{-1}q^{n-1})\dots(1-x^{-1}q)(1-x)\dots(1-xq^{n-1}) = \sum_{l=-n}^{n}\binom{2n}{n+l}_q q^{\frac{l(l-1)}{2}}(-x)^{l}. \tag{12}$$

We now take the limit as $n \to \infty$.

$$\begin{aligned}\binom{2n}{n+l}_q &= \frac{(1-q^{2n})(1-q^{2n-1})\cdots(1-q^{n-l+1})}{(1-q)(1-q^2)\cdots(1-q^{n+l})} \\ &= \frac{\displaystyle\prod_{j=n-l+1}^{2n}(1-q^j)}{\displaystyle\prod_{j=1}^{n+l}(1-q^j)}.\end{aligned}$$

For fixed l and $|q| < 1$, each factor in the numerator tends to 1 as $n \to \infty$, so

$$\lim_{n\to\infty}\prod_{j=n-l+1}^{2n}(1-q^j) = 1.$$

Meanwhile, the denominator tends to the infinite product

$$\lim_{n\to\infty}\prod_{j=1}^{n+l}(1-q^j) = \prod_{j=1}^{\infty}(1-q^j).$$

Therefore,

$$\boxed{\lim_{n\to\infty}\binom{2n}{n+l}_q = \frac{1}{\prod_{j=1}^{\infty}(1-q^j)}}.$$

Substituting this limit into the series and simplifying, we obtain:

$$\prod_{n=1}^{\infty}(1-q^n)(1-xq^{n-1})(1-x^{-1}q^n) = \sum_{l=-\infty}^{\infty} q^{\frac{l(l-1)}{2}}(-x)^l. \tag{13}$$

Finally, substituting $q \mapsto q^2$ and $x \mapsto -xq$, we arrive at the Jacobi Triple Product Identity:

$$\boxed{\prod_{n=1}^{\infty}(1-q^{2n})(1+xq^{2n-1})(1+x^{-1}q^{2n-1}) = \sum_{l=-\infty}^{\infty} x^l q^{l^2}.} \tag{14}$$

$\square$